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Müntz–Jackson Theorems in $L_p(0, 1)$, $1 \leqslant p < 2$

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Let $\Lambda: 0 = \lambda_0 < \lambda_1 < \cdots$ be an infinite sequence of positive numbers, let $n \in \mathbb{N}$ and $B_p(z) := \prod_{k=1}^{n} (z - \lambda_k - 1/p)/(z + \lambda_k + 1/p)$. Ganelius and Newman have shown that the expression $\varepsilon_n(\Lambda)_p = \max_{y \in \mathbb{R}} |B_p(1+iy)/(1+iy)|$ is the approximation index for the error $\mu_n(f, \Lambda)_p := \inf_{b_k} \|f(x) - \sum_{k=0}^{n} b_k x^{\lambda_k}\|_p$ of functions $f \in L_p(0, 1)$ in the L_p-norm on $[0, 1]$, $1 \leqslant p \leqslant \infty$. That is, if f is absolutely continuous on $[0, 1]$, then $\mu_n(f, \Lambda)_p \leqslant A_p \varepsilon_n(\Lambda)_p \|f'\|_p$, where $A_p < 2^{29}$ is a numerical constant. It is the purpose of the present paper to apply another method of proof which produces small factors $A_p < 42$, $1 \leqslant p < 2$. As is well-known, the factor A_p is small if $2 \leqslant p \leqslant \infty$, for example, $A_p < 14$, which has been proved recently by the author.

1. INTRODUCTION

Let $\Lambda: 0 = \lambda_0 < \lambda_1 < \cdots$ be an infinite sequence of positive numbers. Let $\| \; \|_p$ be the L_p-norms on $[0, 1]$. We estimate the error of approximation,

$$\mu_n(f, \Lambda)_p := \inf_{b_k} \left\| f(x) - \sum_{k=0}^{n} b_k x^{\lambda_k} \right\|_p,$$

of functions $f \in L_p(0, 1)$, $1 \leqslant p < 2$.

There are two methods to prove Müntz–Jackson theorems. The first, due to Newman [15], uses a corollary of the Hahn–Banach theorem by which $\mu_n(f, \Lambda)_p$ is characterized as

$$\mu_n(f, \Lambda)_p = \sup_H \int_0^1 f(x) H(x) \, dx, \tag{1.1}$$

where the supremum is taken over all functions $H \in L_q(0, 1)$, $q := p/(p-1)$, satisfying $\|H\|_q = 1$ and

$$\int_0^1 x^{\lambda_k} H(x) \, dx = 0, \qquad k = 0, 1, ..., n.$$

640/67/3-8

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The second method, suggested by the author [7], is more elementary and will be used in this paper: first the function f is approximated on $[0, 1]$ by an appropriate even algebraic polynomial $P_m(x) = \sum_{j=0}^{[m/2]} a_{2j} x^{2j}$, then the monomials x^{2j}, $j = 1, 2, ..., [m/2]$ are replaced by appropriate Λ-polynomials. And we get

$$\mu_n(f, \Lambda)_p \leqslant \|f - P_m\|_p + \sum_{j=1}^{[m/2]} |a_{2j}| \, \mu_n(x^{2j}, \Lambda)_p. \tag{1.2}$$

An essential role will be played by the Blaschke product

$$B_p(z) = \prod_{k=1}^{n} \frac{z - \lambda_k - 1/p}{z + \lambda_k + 1/p}$$

and the number

$$\varepsilon_n(\Lambda)_p = \max_{y \geqslant 0} \left| \frac{B_p(1 + iy)}{1 + iy} \right|. \tag{1.3}$$

For example, (see Feinerman and Newman [4]), in the *separate case*, $\lambda_{k+1} - \lambda_k \geqslant 2$ for $k \geqslant 0$, one has

$$\varepsilon_n(\Lambda)_p \approx \exp\left(-2 \sum_{k=1}^{n} \frac{1}{\lambda_k + 1/p} \right),$$

and in the *unseparate case*, $0 < \lambda_{k+1} - \lambda_k \leqslant 2$, $k = 0, 1, ...,$

$$\varepsilon_n(\Lambda)_p \approx \left(\sum_{k=1}^{n} \left(\lambda_k + \frac{1}{p} \right) \right)^{-1/2}.$$

A general Müntz–Jackson theorem has been established by Ganelius and Newman [5]. They show that the expression $\varepsilon_n(\Lambda)_p$ is the approximation index for the exponents Λ:

THEOREM A. *If* $1 \leqslant p \leqslant \infty$, $n \in \mathbb{N}$, *and if* f *is absolutely continuous on* $[0, 1]$, *then*

$$\mu_n(f, \Lambda)_p \leqslant A_p \varepsilon_n(\Lambda)_p \, \|f'\|_p,$$

where $A_p < 2^{29}$ *is a numerical constant.*

Ganelius and Newman show that this result is the best possible in the sense that it is false for each p, each Λ, and each $n \in \mathbb{N}$ if A_p is replaced by 1/600. Their proofs use the characterization (1.1) and are difficult; their factor $A_p < 2^{29}$, is very large. The method (1.2) is simpler and produces factors $A_p < 42$, $1 \leqslant p < 2$.

It is well-known that A_p is small if $2 \leqslant p \leqslant \infty$; for example, $A_p < 14$, which has been proved recently by the author [11].

2. Approximation of the Monomials

For the L_2-norm on [0, 1] we have the identity

$$\mu_n(x^r, \Lambda)_2 = \frac{1}{\sqrt{2r+1}} \prod_{k=0}^{n} \frac{|r-\lambda_k|}{r+\lambda_k+1},$$

$r > -1/2$. In [11], this has been used to derive the inequality

$$\mu_n(x^r, \Lambda)_p \leqslant \frac{1+1/p}{(2r+2/p)^{1/p}} \prod_{k=1}^{n} \frac{|r-\lambda_k|}{r+\lambda_k+2/p},$$

for $r > -1/p$ and $2 < p \leqslant \infty$, in particular for the uniform norm on [0, 1], for $r > 0$,

$$\mu_n(x^r, \Lambda)_\infty \leqslant \prod_{k=1}^{n} \frac{|r-\lambda_k|}{r+\lambda_k},$$

which has been derived first in [7].

Similar results for $1 \leqslant p < 2$ are more difficult to get. The following inequality is the main new achievement of this paper:

Lemma 2.1. *For $1 \leqslant p < 2$ and any real number $r \geqslant 2$ one has*

$$\mu_n(x^r, \Lambda)_p \leqslant 2^{1/p}\{2(r+1/p)\}^{r+1}\, \varepsilon_n(\Lambda)_p^{r+1/p}. \tag{2.1}$$

Proof. We set

$$l_k := \frac{\lambda_k + 1/p}{r+1/p}, \qquad B(z) := \prod_{k=1}^{n} \frac{z-l_k}{z+l_k}, \qquad u(z) := \frac{1}{z+1},$$

and $F(z) := u(z)\, B(z)$. F is of the form

$$F(z) = \frac{B(-1)}{z+1} - \sum_{k=1}^{n} \frac{c_k}{z+l_k}$$

with some real coefficients c_k. The evaluation of the integrals (or the standard residue argument) gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ity}}{iy+\rho}\, dy = \begin{cases} e^{-\rho t} & \text{if } \ t>0 \\ 0, & \text{if } \ t<0 \end{cases}$$

for all positive numbers ρ. Hence the inverse Fourier transform

$$h(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(iy)\, e^{ity}\, dy$$

of $F(iy)$ satisfies $h(t)=0$ if $t<0$ and is of the form

$$h(t)=B(-1)\, e^{-t}-\sum_{k=1}^{n} c_k e^{-tl_k}, \qquad t>0.$$

We set $b_k=c_k/B(-1)$. Substituting $x=\exp(-t/(r+1/p))$ we get

$$\mu_n(x^r, \Lambda)_p \leqslant \left\| x^r-\sum_{k=1}^{n} b_k x^{\lambda_k}\right\|_p \leqslant (r+1/p)^{-1/p}\, |B(-1)|^{-1}\, \|h\|_{L_p(0,\infty)}. \tag{2.2}$$

The difficult part is to estimate $\|h\|_{L_p(0,\infty)}$. We note that

$$\int_{-\infty}^{\infty} |u'(iy)|^2\, dy=\int_{-\infty}^{\infty} \frac{dy}{(1+y^2)^2}=\frac{\pi}{2}.$$

We set $c:=2\sum_{k=1}^{n} l_k^{-1}$ and

$$S_n:=\sum_{k=1}^{n} l_k^{-3/2}=(r+1/p)^{3/2}\sum_{k=1}^{n} (\lambda_k+1/p)^{-3/2}.$$

Then

$$\left|\frac{B'(iy)}{B(iy)}+c\right|=\sum_{k=1}^{n} \frac{2y^2}{l_k(y^2+l_k^2)}$$

and

$$\int_{-\infty}^{\infty} |u(iy)|^2 \left|c+\frac{B'(iy)}{B(iy)}\right|^2 dy \leqslant \int_{-\infty}^{\infty} \left(\sum_{k=1}^{n} \frac{2y}{l_k(y^2+l_k^2)}\right)^2 dy$$
$$\leqslant \left(\sum_{k=1}^{n} \sqrt{\int_{-\infty}^{\infty} \frac{4y^2\, dy}{l_k^2(y^2+l_k^2)^2}}\right)^2=2\pi S_n^2.$$

For a fixed number a, $a\geqslant 1/\pi$, and $v(t):=a^2+(t-c)^2$ we apply Hölder's inequality for the exponent $q:=2/p$. Then

$$\|h\|_{L_p(0,\infty)}^2=\left(\int_0^{\infty} v(t)^{-p/2}\, |\sqrt{v(t)}\, h(t)|^p\, dt\right)^{2/p} \leqslant K\int_0^{\infty} v(t)\, |h(t)|^2\, dt,$$

where

$$K:=\left(\int_0^{\infty} v(t)^{-p/(2-p)}\, dt\right)^{-1+2/p}.$$

Since $p/(2-p) \geqslant 1$ and $a\pi \geqslant 1$,

$$K \leqslant a^{-3+2/p} \left(\int_{-\infty}^{\infty} (1+t^2)^{-p/(2-p)}\, dt \right)^{-1+2/p} \leqslant \frac{\pi}{a}.$$

This proves that

$$\|h\|^2_{L_p(0,\infty)} \leqslant \pi \left(a \int_0^{\infty} |h(t)|^2\, dt + \frac{1}{a} \int_0^{\infty} (t-c)^2\, |h(t)|^2\, dt \right). \tag{2.3}$$

By Parseval's identity,

$$\int_0^{\infty} |h(t)|^2\, dt = \int_{-\infty}^{\infty} |h(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(iy)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |u(iy)|^2\, dt,$$

where we have used that $|B(iy)| = 1$ for any real y. Hence,

$$\int_0^{\infty} |h(t)|^2\, dt = \frac{1}{2}. \tag{2.4}$$

Since $F(iy) \to 0$ as $y \to \pm\infty$, integration by parts leads to

$$2\pi \int_{-\infty}^{\infty} (t-c)^2\, |h(t)|^2\, dt = \int_{-\infty}^{\infty} \left| \frac{d}{dy} (F(iy)\, e^{icy}) \right|^2 dy.$$

From $|B(iy)| = 1$, $y \in \mathbb{R}$, we get

$$\left| \frac{d}{dy} (F(iy)\, e^{icy}) \right| \leqslant |u'(iy)| + |u(iy)| \left| c + \frac{B'(iy)}{B(iy)} \right|$$

and therefore

$$\sqrt{2\pi}\, \|(t-c)\, h(t)\|_{L_2(0,\infty)} \leqslant \|u'(iy)\|_{L_2(\mathbb{R})} + \left\| u(iy) \left(c + \frac{B'(iy)}{B(iy)} \right) \right\|_{L_2(\mathbb{R})},$$

so that

$$\|(t-c)\, h(t)\|_{L_2(0,\infty)} \leqslant \tfrac{1}{2} + S_n. \tag{2.5}$$

Inserting this and (2.4) into (2.3), with $a := \sqrt{2}\,(S_n + 1/2)$, yields

$$\|h\|^2_{L_p(0,\infty)} \leqslant \sqrt{2}\, \pi (S_n + 1/2). \tag{2.6}$$

Introducing the notation

$$\delta_n := \sum_{k=1}^{n} (\lambda_k + 1/p)^{-3/2} = (r + 1/p)^{-3/2} S_n$$

it follows from (2.2) and (2.6) and the inequality $2^{1/4}(r+1/p)^{-1/4} \leqslant 1$, $r \geqslant 2$, $1 \leqslant p < 2$, that

$$\mu_n(x^r, \Lambda)_p \leqslant \sqrt{\pi}\,(r+1/p)^{1-1/p}\,|B(-1)|^{-1}\,(\sqrt{\delta_n} + 1/2). \tag{2.7}$$

By the definition of $B(-1)$ and an inequality of Newman [16], we have

$$|B(-1)|^{-1} = \prod_{k=1}^{n} \frac{|r-\lambda_k|}{r+\lambda_k+2/p} \leqslant ((r+1/p)\,\varepsilon_n(\Lambda)_p)^{r+1/p}. \tag{2.8}$$

If $\delta_n \leqslant 4^{r+1/p}$, then (2.1) follows by (2.7) and (2.8) and the observation that $\sqrt{\pi}\,(2^{r+1/p} + 1/2) \leqslant 2^{r+1+1/p}$. Otherwise we apply the next lemma. ∎

LEMMA 2.2. *If $r \geqslant 2$, $1 \leqslant p < 2$, and if Λ is a sequence of positive numbers satisfying $\delta_n > 4^{r+1/p}$, then*

$$\sqrt{\delta_n} \prod_{k=1}^{n} \frac{|r-\lambda_k|}{r+\lambda_k+2/p} \leqslant \{2(r+1/p)\,\varepsilon_n(\Lambda)_p\}^{r+1/p}. \tag{2.9}$$

Proof. We set $\rho := r + 1/p \geqslant 5/2$, $s_k := \lambda_k + 1/p$, and define the functions

$$G(s, y) = \left(\frac{s-\rho}{s+\rho}\right)^2 \left(\frac{y^2+(s+1)^2}{y^2+(s-1)^2}\right)^{\rho}, \qquad H(y) = \sum_{k=1}^{n} \log G(s_k, y).$$

For fixed $s > 0$, $G(s, y)$ is a monotone decreasing function in $0 \leqslant y < \infty$, hence $H(y)$ is also monotone decreasing for $y \geqslant 0$. In addition, the logarithmic derivative of $G(s, 0)$ is $4\rho(\rho^2-1)/((s^2-\rho^2)(s^2-1))$. Hence $G(s, 0)$ is monotone increasing in $\rho < s < \infty$ and

$$G(s, y) \leqslant G(s, 0) = \frac{(s-\rho)^2}{(s+\rho)^2} \frac{(s+1)^{2\rho}}{(s-1)^{2\rho}} < G(+\infty, 0) = 1, \qquad s > \rho. \tag{2.10}$$

Let $\rho < s \leqslant y/2$. Using the inequality $(1-x)/(1+x) \leqslant e^{-2x}$, $0 < x < 1$, we then obtain

$$\log G(s, y) \leqslant -\frac{4\rho}{s} + \rho \log\left\{1 + \frac{4s}{y^2+(s-1)^2}\right\} \leqslant -\frac{4\rho}{s} + \frac{4\rho s}{y^2} \leqslant -\frac{3\rho}{s}.$$

Similarly, if $s < \rho \leqslant y/2$, then $\log G(s, y) \leqslant -3s/\rho$.

We define the sets of indices

$$M_j := \{k \in \mathbb{N} : 2^{j-1}\rho \leqslant s_k < 2^j \rho\}, \qquad j \geqslant 1,$$

and get from (2.10) that

$$H(2\rho) \leqslant \sum_{s_k < \rho} \log G(s_k, 2\rho),$$

$$H(2^{j+1}\rho) \leqslant \sum_{k \in M_j} \log G(s_k, 2^{j+1}\rho), \qquad j \geqslant 1.$$

Since $s_k > 1/p > 1/2$ and $p^{5/2} \leqslant 6$ it follows that

$$H(2\rho) \leqslant -\frac{3}{\rho} \sum_{s_k < \rho} s_k \leqslant -\frac{1}{2\rho} \sum_{s_k < \rho} s_k^{-3/2}, \tag{2.11}$$

and for $j = 1, 2, \ldots,$

$$H(2^{j+1}\rho) \leqslant -3\rho \sum_{k \in M_j} s_k^{-1} \leqslant -3\rho 2^{j/2} \sum_{k \in M_j} s_k^{-3/2}. \tag{2.12}$$

Hence, taking the sums of (2.11) and (2.12),

$$\begin{aligned} \delta_n &= \sum_{s_k < \rho} s_k^{-3/2} + \sum_{j=1}^{\infty} \sum_{k \in M_j} s_k^{-3/2} \\ &\leqslant -2\rho H(2\rho) - \frac{1}{3\rho} \sum_{j=1}^{\infty} 2^{-j/2} H(2^{j+1}\rho). \end{aligned} \tag{2.13}$$

By the definition of $\varepsilon_n(\Lambda)_p$ as the maximum (1.3), it follows that (2.9) is valid if and only if

$$H(y) \leqslant -\log \delta_n - \rho \log\{(1 + y^2)/(4\rho^2)\} \tag{2.14}$$

holds for at least one $y \geqslant 0$.

Let us suppose to the contrary that (2.9) is wrong, hence that (2.14) is wrong for all $y \geqslant 0$. Then we have from (2.13) that

$$\begin{aligned} \delta_n &\leqslant 2\rho \log \delta_n + 2\rho^2 \log(1 + 1/(4\rho^2)) \\ &\quad + \frac{1}{3\rho} \sum_{j=1}^{\infty} 2^{-j/2} \{\log \delta_n + \rho \log(4^j + 1/(4\rho^2))\}, \end{aligned}$$

hence

$$\delta_n \leqslant (2\rho + 1) \log \delta_n + 4\rho, \tag{2.15}$$

if we use that $\log \delta_n \geqslant \rho \log 4$, $\rho \geqslant 5/2$ and

$$\sum_{j=1}^{\infty} 2^{-j/2} \log(4^j + 1/(4\rho^2)) \leqslant 12.$$

But the inequalities (2.15) and $\delta_n \geqslant 4^\rho$ cannot be valid simultaneously, a contradiction. ∎

3. Müntz–Jackson Theorems

We shall need

LEMMA 3.1. *For $1 \leqslant p \leqslant 2$, $m \geqslant 1$, and absolutely continuous functions f on $[0, 1]$ there exists an even algebraic polynomial $P_m(x) = \sum_{j=0}^{[m/2]} a_{2j} x^{2j}$ for which*

$$\|f - P_m\|_p \leqslant \frac{\sqrt{2}\,\pi}{2(m+1)} \|f'\|_p, \tag{3.1}$$

$$|a_{2j}| \leqslant K_p m^{2j-1+1/p} \|f'\|_p/(2j)!, \qquad 1 \leqslant j \leqslant [m/2], \tag{3.2}$$

where $K_p \leqslant \sqrt{2\pi}\,(2\sqrt{2})^{1/p}$.

The proof of the last lemma can be found in [11], also for $2 < p \leqslant \infty$ with $K_p \leqslant 2\pi$.

We shall now prove our main result:

THEOREM 3.2. *For $1 \leqslant p < 2$, the factor A_p in Theorem* A *is less than* 42.

Proof. Set $h = \varepsilon_n(\Lambda)_p$ and define the integer m by

$$hm \leqslant 1/12 < h(m+1).$$

Since $0 \in \Lambda$ and $\|f - f(0)\|_p \leqslant \|f'\|_p$ we may assume that $h < 1/42$ and thus that $m \geqslant 3$. We insert Lemma 2.1 and Lemma 3.1 into (1.2). This yields that $\mu_n(f, \Lambda)_p$ is

$$\leqslant \left\{ \frac{\sqrt{2}\,\pi}{2h(m+1)} + 2^{1/p} K_p \sum_{j=1}^{[m/2]} \frac{\{2(2j+1/p)\}^{2j+1}}{(2j)!} (mh)^{2j-1+1/p} \right\} h \|f'\|_p$$

$$\leqslant \left\{ 6\pi\sqrt{2} + 24 K_p 6^{-1/p} \sum_{j=1}^{\infty} \frac{(2j+1/p)^{2j+1}\, 6^{-2j}}{(2j)!} \right\} h \|f'\|_p.$$

Since $K_p \leqslant \sqrt{2\pi}\,(2\sqrt{2})^{1/p}$, it is easy to confirm that the expression in the brackets is less than 42, for all $1 \leqslant p < 2$. ∎

Let the smoothness of functions $f \in L_p(0, 1)$ be measured by the Lebesgue modulus

$$\omega_p(f; \delta) := \sup_{0 < h \leqslant \delta} \left(\int_0^{1-h} |f(x+h) - f(x)|^p \, dx \right)^{1/p};$$

let $\omega_\infty(f; \delta)$ be the usual modulus of continuity.

From Theorem A (and Theorem 3.2) one obtains, by standard techniques, Müntz–Jackson theorems for other function classes in $L_p(0, 1)$:

THEOREM 3.3. *Let* $r = 0, 1, \ldots$ *and let* $\lambda_k = k$ *for* $k = 0, \ldots, r$. *If* $f^{(r)} \in L_p(0, 1)$, $1 \leqslant p < \infty$, *or if* $f^{(r)} \in C[0, 1]$, $p = \infty$, *then, for* $n \geqslant r+1$,

$$\mu_n(f, \Lambda)_p \leqslant C_{r,p} \varepsilon_n(\Lambda)_p^r \, \omega_p(f^{(r)}; \varepsilon_n(\Lambda)_p),$$

where $C_{r,p}$ *is independent of* f *and* n.

For example, if $r = 0$, one has $C_{0,p} < 84$, $1 \leqslant p < 2$. This follows from Theorem 3.2 and Proposition 2.1 in [5]. A detailed representation of the theory of Müntz polynomials will be given in Lorentz, v. Golitschek, and Makovoz [14] including a complete proof of Theorem 3.3, but also Müntz–Jackson theorems for positive intervals $[a, b]$, $0 < a < b$.

REFERENCES

1. J. BAK AND D. J. NEWMAN, Müntz–Jackson theorems in $L_p(0, 1)$ and $C[0, 1]$, *Amer. J. Math.* **94** (1972), 437–457.
2. J. BAK, D. LEVIATAN, D. J. NEWMAN, AND J. TZIMBALARIO, Generalized polynomial approximation, *Israel J. Math.* **15** (1973), 337–349.
3. J. BAK AND D. J. NEWMAN, Müntz–Jackson theorems in L_p, $p < 2$, *J. Approx. Theor.* **10** (1974), 218–226.
4. R. P. FEINERMAN AND D. J. NEWMAN, "Polynomial Approximation," Williams & Wilkins, Baltimore MD, 1974.
5. T. H. GANELIUS AND D. J. NEWMAN, Müntz–Jackson theorems in L_p-spaces with unrestricted exponents, *Amer. J. Math.* **98** (1976), 295–309.
6. T. H. GANELIUS AND S. WESTLUND, The degree of approximation in Müntz-theorem, *in* "Proceedings of the Conference on Mathematical Analysis, Jyraskya, Finland, August 1970."
7. M. V. GOLITSCHEK, Erweiterung der Approximationssätze von Jackson im Sinne von C. Müntz, *J. Approx. Theory* **3** (1970), 72–86.
8. M. V. GOLITSCHEK, Jackson–Müntz Sätze in der L_p-Norm, *J. Approx. Theory* **7** (1973), 87–106.
9. M. V. GOLITSCHEK, Jackson–Müntz theorems in $L_p(0, 1)$ and $C[0, 1]$ for complex exponents, *J. Approx. Theory* **18** (1976), 13–29.
10. M. V. GOLITSCHEK, Lineare Approximation durch komplexe Exponentialsummen, *Math. Z.* **146** (1976), 17–32.

11. M. v. GOLITSCHEK, Müntz polynomials and approximation with weights, *in* "Approximation Theory VI" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), Vol. I, Academic Press, San Diego, 1989.
12. D. LEVIATAN, On the Jackson–Müntz theorem, *J. Approx. Theory* **10** (1974), 1–5.
13. D. LEVIATAN, On the rate of approximation by polynomials with complex exponents, *J. London Math. Soc.* **15** (1977), 305–318.
14. G. G. LORENTZ, M. v. GOLITSCHEK, AND Y. MAKOVOZ, "Constructive Approximation. Advanced Problems," Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin/New York, to appear.
15. D. J. NEWMAN, A Müntz–Jackson theorem, *Amer. J. Math.* **87** (1965), 940–944.
16. D. J. NEWMAN, A review of Müntz–Jackson theorems, *in* "Approximation Theory" (G. G. Lorentz, Ed.), Academic Press, San Diego, 1973.
17. D. J. NEWMAN, The Müntz–Jackson theorem in L_2, *J. Approx. Theory* **9** (1973), 91–95.
18. D. J. NEWMAN, A general Müntz–Jackson theorem, *Amer. J. Math.* **96** (1974), 340–345.